Fall 2013 **Math 601** Name: ______________________

Quiz 10

Question 1. (10 pts)
Evaluate the integral

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz$$

(a) when C is the circle $|z-3| = 1$, that is, the circle centered at 3 with radius 1.

Solution: Notice that

$$\frac{e^z}{z^2 - 4z + 3} = \frac{e^z}{(z-1)(z-3)}$$

So $z = 3$ is inside the circle $|z-3| = 1$ and $z = 1$ is outside the circle $|z-3| = 1$. By Cauchy's formula, we have

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \int_C \frac{e^z/(z-1)}{z-3} dz = 2\pi i \frac{e^3}{3-1} = e^3 \pi i.$$

(b) when C is the circle $|z| = 2$, i.e., the circle centered at 0 with radius 2.

Solution: This time, $z = 1$ is inside the circle $|z| = 2$ and $z = 3$ is outside the circle $|z| = 2$. By Cauchy's formula, we have

$$\int_C \frac{e^z}{z^2 - 4z + 3} dz = \int_C \frac{e^z/(z-3)}{z-1} dz = 2\pi i \frac{e}{1-3} = -e\pi i.$$

Question 2. (10 pts)

Show that

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz = \sin t$$

where C is the the circle $|z| = 2$.

Solution:

$$\frac{e^{zt}}{z^2+1} = \frac{e^{zt}}{(z+i)(z-i)}$$

Both i and $-i$ lie inside the circle $|z| = 3$. Consider the partial fractions

$$\frac{1}{(z+i)(z-i)} = \frac{A}{z-i} - \frac{B}{z+i} = \frac{A(z+i) - B(z-i)}{(z+i)(z-i)}$$

So $A = B = \frac{1}{2i}$. Now apply Cauchy's formula

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2+1}dz \\
&= \frac{1}{2\pi i}\int_C \frac{e^{zt}}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz \\
&= \frac{e^{it}}{2i} - \frac{e^{-it}}{2i} \\
&= \frac{\cos t + i\sin t}{2i} - \frac{\cos t - i\sin t}{2i} \\
&= \sin t
\end{aligned}$$